Defects, Rigid Holography and C-Theorems

Ignacio Carreño Bolla
Universidad de Oviedo and ICTEA

Based on:

ICB, D. Rodriguez-Gomez, J. Russo [2303.01935,2306.11796]

Jornadas ICTEA, Oviedo, 6^{th}-7^{th} May 2024

University of Oviedo

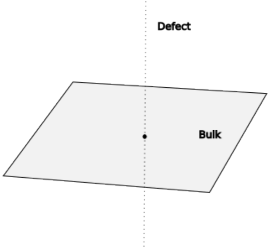
ICTEA

Outline

1. Introduction to Defect CFTs and Renormalization Group
2. RG Flows in Defect CFTs
3. Rigid Holography and C-Theorems

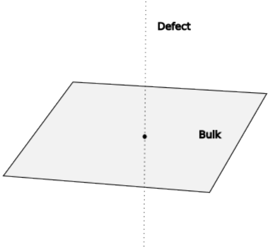
What is a defect?

- QFTs are defined on a $d-$dimensional spacetime M_d (**bulk**), but we can restrict degrees of freedom to a subspace of dimension $q < p$ $N_q \subset M_d$ (**defect**). This is a **defect QFT**.



What is a defect?

- QFTs are defined on a $d-$dimensional spacetime M_d (**bulk**), but we can restrict degrees of freedom to a subspace of dimension $q < p$ $N_q \subset M_d$ (**defect**). This is a **defect QFT**.



- Defects are ubiquitous!: impurities in CM, Wilson loops in QCD, even symmetry operators= topological defects!

Renormalization Group and C-Theorem

- Physics depends on scale μ! We can know the theory at low energy (**IR**), by "averaging out" the highest energy modes from the original theory (**UV**). This is **Renormalization Group (RG) flow**, given by β-functions.

$$\beta = \mu \frac{\partial g}{\partial \mu} \tag{1}$$

Renormalization Group and C-Theorem

- Physics depends on scale μ! We can know the theory at low energy (**IR**), by "averaging out" the highest energy modes from the original theory (**UV**). This is **Renormalization Group (RG) flow**, given by β-functions.

$$\beta = \mu \frac{\partial g}{\partial \mu} \tag{1}$$

- The number of d.o.f. is lower in the IR than in the UV (RG flow is irreversible): C-**theorem**.

$$C = C(g), \quad C_{UV} - C_{IR} > 0 \tag{2}$$

Defect CFTs

- The flow may reach a fixed point ($\beta = 0$), the theory does not depend on scale (**CFT**). We may perturb a CFT by "turning on" a **deformation** that lives on the defect.

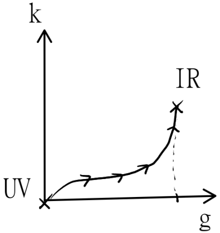
$$S = S_{CFT} + g \int_{N_q} \mathcal{O}_{def} \tag{3}$$

Defect CFTs

- The flow may reach a fixed point ($\beta = 0$), the theory does not depend on scale (**CFT**). We may perturb a CFT by "turning on" a **deformation** that lives on the defect.

$$S = S_{CFT} + g \int_{N_q} \mathcal{O}_{def} \tag{3}$$

- This deformation only results in a new flow, if it is **relevant**, i.e. $\Delta(g) > 0$ or **marginal** $\Delta(g) = 0$ (quantum corrections). This new flow may end end on a new fixed point $g = g_*$.



Scalar CFTs

- We want to study scalar theories with N scalars Φ_i in $d = 4, 6$ dimensional bulks M_d, given by

$$S_{bulk} = \int_{M_d} \frac{1}{2}(\partial_\mu \Phi_i)^2 + V(\Phi_i) \tag{4}$$

with $V(\Phi_i)$ a homogeneous polynomial of degree $n = 4, 3$ respectively with marginal coupling g_α. We let the bulk flow to a CFT.

Scalar CFTs

- We want to study scalar theories with N scalars Φ_i in $d = 4, 6$ dimensional bulks M_d, given by

$$S_{bulk} = \int_{M_d} \frac{1}{2}(\partial_\mu \Phi_i)^2 + V(\Phi_i) \tag{4}$$

with $V(\Phi_i)$ a homogeneous polynomial of degree $n = 4, 3$ respectively with marginal coupling g_α. We let the bulk flow to a CFT.

- We perturb this CFT by adding a deformation located in a $q = 1, 2$ dimensional flat defect N_q.

$$S = S_{bulk} + S_{defect}, \quad S_{defect} = -\int_{N_q} h_i \Phi_i \tag{5}$$

Scalar CFTs

- We want to study scalar theories with N scalars Φ_i in $d = 4, 6$ dimensional bulks M_d, given by

$$S_{bulk} = \int_{M_d} \frac{1}{2}(\partial_\mu \Phi_i)^2 + V(\Phi_i) \tag{4}$$

with $V(\Phi_i)$ a homogeneous polynomial of degree $n = 4, 3$ respectively with marginal coupling g_α. We let the bulk flow to a CFT.

- We perturb this CFT by adding a deformation located in a $q = 1, 2$ dimensional flat defect N_q.

$$S = S_{bulk} + S_{defect}, \quad S_{defect} = -\int_{N_q} h_i \Phi_i \tag{5}$$

- The defect acts as a "source" for ϕ:

$$\langle \phi_i \rangle \sim \frac{h_i}{r^\Delta} \tag{6}$$

Double Scaling Limit

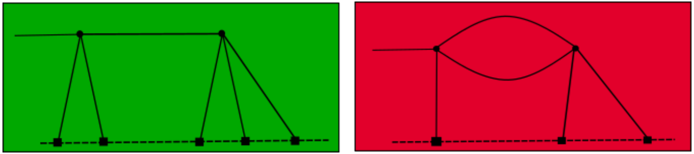
- We can try to compute $\beta_i = \mu \partial h_i / \partial \mu$, but this is difficult because of bulk loops!

Double Scaling Limit

- We can try to compute $\beta_i = \mu \partial h_i / \partial \mu$, but this is difficult because of bulk loops!
- We take the semiclassical **double scaling limit**

$$h_i \to \infty, \; g_\alpha \to 0, \; g_\alpha h_i^{n-2} \text{ fixed} \tag{7}$$

This suppresses bulk loops!

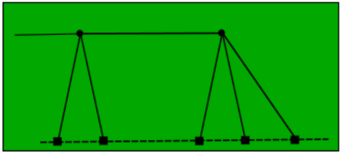


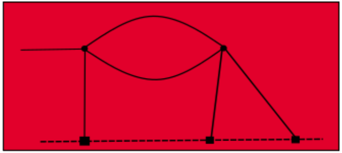
Double Scaling Limit

- We can try to compute $\beta_i = \mu \partial h_i / \partial \mu$, but this is difficult because of bulk loops!
- We take the semiclassical **double scaling limit**

$$h_i \to \infty, \; g_\alpha \to 0, \; g_\alpha h_i^{n-2} \text{ fixed} \tag{7}$$

This suppresses bulk loops!





- We compute β_i via **dimensional regulation** $(d' = d - \epsilon, \; \epsilon \to 0)$

Remarks on the β-functions

- The fixed points of β show factorization of the dependence of ϵ with g_α (determined at 1-loop). This is **dimensional disentanglement**.

Remarks on the β-functions

- The fixed points of β show factorization of the dependence of ϵ with g_α (determined at 1-loop). This is **dimensional disentanglement**.
- Under a change of scheme,

$$\beta = 2c \frac{\partial \mathcal{H}(\phi_i)}{\partial \phi_i} \tag{8}$$

for a scalar function $\mathcal{H}(\phi_i)$ that monotonically decreases along RG flow.

Instabilities in Defect CFTs

- Our defect CFT may become unstable if at some value of the couplings, operators become relevant!

Instabilities in Defect CFTs

- Our defect CFT may become unstable if at some value of the couplings, operators become relevant!
- We look for condensates $\mathcal{O} = \phi_i^k$. Due to the semiclassical limit $< \mathcal{O} >=< \phi_i >^k$ (as in AdS/CFT).

Instabilities in Defect CFTs

- Our defect CFT may become unstable if at some value of the couplings, operators become relevant!
- We look for condensates $\mathcal{O} = \phi_i^k$. Due to the semiclassical limit $< \mathcal{O} >=< \phi_i >^k$ (as in AdS/CFT).
- Therefore $\Delta(\mathcal{O}) = k\Delta(\phi_i)$, and we read this off the one point functions. If $\Delta(\mathcal{O}) < q$, the defect CFT becomes unstable.

Instabilities in Defect CFTs

- Our defect CFT may become unstable if at some value of the couplings, operators become relevant!
- We look for condensates $\mathcal{O} = \phi_i^k$. Due to the semiclassical limit $< \mathcal{O} > = < \phi_i >^k$ (as in AdS/CFT).
- Therefore $\Delta(\mathcal{O}) = k\Delta(\phi_i)$, and we read this off the one point functions. If $\Delta(\mathcal{O}) < q$, the defect CFT becomes unstable.
- We check for instabilities in scalar theories with $O(N)$ symmetry and fermionic Yukawa models.

Defect Free Energy

- Can we define a C-theorem for our defect CFT? A candidate is the **defect free energy** on $N_q = \mathbb{S}^q$

$$\mathcal{F}[\mathbb{S}^q] = -\log \frac{\int \mathcal{D}\phi \; e^{-S_{bulk+defect}[\mathbb{S}^d]}}{\int \mathcal{D}\phi \; e^{-S_{bulk}[\mathbb{S}^d]}} \tag{9}$$

Defect Free Energy

- Can we define a C-theorem for our defect CFT? A candidate is the **defect free energy** on $N_q = \mathbb{S}^q$

$$\mathcal{F}[\mathbb{S}^q] = -\log \frac{\int \mathcal{D}\phi \; e^{-S_{bulk+defect}[\mathbb{S}^d]}}{\int \mathcal{D}\phi \; e^{-S_{bulk}[\mathbb{S}^d]}} \tag{9}$$

- Expanding this in term of the radius R:

$$\mathcal{F}_q = \begin{cases} c^{(1)}\,(\Lambda R) - s_1 & q = 1\,, \\ c^{(2)}(\Lambda R)^2 + c^{(0)} - s_2 \log(\Lambda R) & q = 2\,. \end{cases} \tag{10}$$

where only s_d is **universal** and is **monotonically decreasing along RG flows**

Computing $\mathcal{F}$ for Defect CFT

- We perform a conformal transformation on the bulk to $M_d = \mathbb{H}^{q+1} \times \mathbb{S}^{q+1}$ (note no conformal coupling to curvature!). The defect lives at $\partial\mathbb{H}^{q+1} = \mathbb{S}^q$ (**rigid holography**).

Computing $\mathcal{F}$ for Defect CFT

- We perform a conformal transformation on the bulk to $M_d = \mathbb{H}^{q+1} \times \mathbb{S}^{q+1}$ (note no conformal coupling to curvature!). The defect lives at $\partial\mathbb{H}^{q+1} = \mathbb{S}^q$ (**rigid holography**).
- Due to the semiclassical limit we can directly compute $\int \mathcal{D}\phi \; e^{-S} = e^{-S_{on-shell}}$ via saddle-point approximation.

Computing $\mathcal{F}$ for Defect CFT

- We perform a conformal transformation on the bulk to $M_d = \mathbb{H}^{q+1} \times \mathbb{S}^{q+1}$ (note no conformal coupling to curvature!). The defect lives at $\partial\mathbb{H}^{q+1} = \mathbb{S}^q$ (**rigid holography**).
- Due to the semiclassical limit we can directly compute $\int \mathcal{D}\phi\, e^{-S} = e^{-S_{on-shell}}$ via saddle-point approximation.
- The action is divergent since the volume of $\mathbb{H}^{q+1}$ is infinite. We regulate this by putting the boundary at finite $r = R$.

Computing $\mathcal{F}$ for Defect CFT

- We perform a conformal transformation on the bulk to $M_d = \mathbb{H}^{q+1} \times \mathbb{S}^{q+1}$ (note no conformal coupling to curvature!). The defect lives at $\partial\mathbb{H}^{q+1} = \mathbb{S}^q$ (**rigid holography**).
- Due to the semiclassical limit we can directly compute $\int \mathcal{D}\phi\, e^{-S} = e^{-S_{on-shell}}$ via saddle-point approximation.
- The action is divergent since the volume of $\mathbb{H}^{q+1}$ is infinite. We regulate this by putting the boundary at finite $r = R$.
- We solve the e.o.m of ϕ by imposing the boundary condition

$$\phi|_\partial = \left(\frac{q}{4\pi}\right)^q h_i \tag{11}$$

which sources ϕ from the defect.

Hamilton-Jacobi Equations

- Alternatively, we can use the **Hamilton-Jacobi equations** identifying "time" with the UV regulator R

$$H(\phi_i, p_i, R) = -\frac{\partial S_{on-shell}}{\partial R} \tag{12}$$

Hamilton-Jacobi Equations

- Alternatively, we can use the **Hamilton-Jacobi equations** identifying "time" with the UV regulator R

$$H(\phi_i, p_i, R) = -\frac{\partial S_{on-shell}}{\partial R} \tag{12}$$

- We find after imposing the b.c. on the boundary:

$$S_{on-shell} = \pi^{1-q} v(R) \mathcal{H}(h_i) \tag{13}$$

This $\mathcal{H}$ is the same function we obtained from field-theory!

Hamilton-Jacobi Equations

- Alternatively, we can use the **Hamilton-Jacobi equations** identifying "time" with the UV regulator R

$$H(\phi_i, p_i, R) = -\frac{\partial S_{on-shell}}{\partial R} \tag{12}$$

- We find after imposing the b.c. on the boundary:

$$S_{on-shell} = \pi^{1-q} v(R) \mathcal{H}(h_i) \tag{13}$$

  This $\mathcal{H}$ is the same function we obtained from field-theory!

- Moreover, using $p_i = \partial S_{on-shell}/\partial \phi$ we recover at the $R \to \infty$ limit:

$$\beta = \frac{2}{q} \frac{\partial \mathcal{H}}{\partial \phi} \tag{14}$$

C-Theorem

- We recover the entropy functions s_q by reading off the on-shell action:

$$s_1 = \mathcal{H}, \quad s_2 = \frac{\mathcal{H}}{2\pi} \tag{15}$$

C-Theorem

- We recover the entropy functions s_q by reading off the on-shell action:

$$s_1 = \mathcal{H}, \quad s_2 = \frac{\mathcal{H}}{2\pi} \tag{15}$$

- They are monotonically decreasing since if $t = -\log \mu$

$$\frac{ds_q}{dt} = -\frac{\beta_i \beta^i}{2\pi^{q-1}} \leq 0 \tag{16}$$

We have a (semiclassical) C-theorem!

C-Theorem

- We recover the entropy functions s_q by reading off the on-shell action:

$$s_1 = \mathcal{H}, \quad s_2 = \frac{\mathcal{H}}{2\pi} \tag{15}$$

- They are monotonically decreasing since if $t = -\log \mu$

$$\frac{ds_q}{dt} = -\frac{\beta_i \beta^i}{2\pi^{q-1}} \leq 0 \tag{16}$$

  We have a (semiclassical) C-theorem!

- Going beyond semiclassics, we check that up to 1-loop in $d = 4 - \epsilon$ line defect scalar QFT, β is a gradient. We find a condition for the gradient property for fermionic line defects in $d = 4 - \epsilon$.

Thank you for your attention!